

EFFECTIVE ACTIONS FOR REGGE STATE-SUM MODELS OF QUANTUM GRAVITY

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We study the semiclassical expansion of the effective action for a Regge state-sum model and its dependence on the choice of the path-integral measure and the spectrum of the edge lengths. If the positivity of the edge lengths is imposed in the effective action equation, we find that the semiclassical expansion is not possible for the power-law measures, while the exponential measures allow the semiclassical expansion. Furthermore, a slightly generalized exponential measure can generate the cosmological term in the effective action as a quantum correction, with a naturally small value of the cosmological constant. We also find that in the case of a discrete length spectrum, the semiclassical expansion is allowed only if the spectrum gap is much smaller than Planck length.

1. Introduction

The idea to define the path integral for General Relativity (GR) by using the spacetime triangulations started with the Regge calculus, see [1, 2] for a review and references. In this way, the path integral (PI) for GR becomes a finite-dimensional integral, and the main problem is how to define this integral in the smooth spacetime limit, which corresponds to an infinite-dimensional integral. However, the development of the spin foam (SF) models of quantum gravity, see [3] for a review and references, reinforced the idea

that the spacetime may be fundamentally discrete. Hence one can consider a quantum gravity theory where the simplicial lattice is physically important, and not only an auxiliary tool. If we apply this way of thinking about QG to quantum Regge calculus, then the nature of the smooth limit problem will be changed. Namely, if we assume that the spacetime is described by a piece-wise linear (PL) 4-manifold, then what we perceive at large distances as a smooth manifold, would be simply a PL manifold with a large number of cells. Therefore, instead of trying to define an infinite-dimensional integral, we would only need to study the large-dimension asymptotics of a finite-dimensional integral.

We will then apply the techniques developed for the SF models and related state sum (SS) models to the Regge discretization of GR. Spin foam models are special cases of the state sum models, where the idea is to define the path integral as a sum over the labels of an amplitude for a labeled triangulation. This idea comes from the topological quantum field theory (TQFT), where the amplitude depends only on the topological class of the triangulation, so that the state sum is independent of the triangulation refinement. Hence the limit of an infinitely refined triangulation is trivially computed in TQFT. However, in the case of a QG state sum, the amplitude depends on a triangulation, and this creates a conflict with the diffeomorphism invariance of the smooth-manifold limit. One can try to take the infinite refinement limit or sum over the triangulations, as in the CDT approach [4], but both approaches are mathematically difficult. Alternatively, if the spacetime is fundamentally discrete, then one can approximate a smooth manifold with a triangulation whose number of cells is sufficiently big. This also means that the diffeomorphism invariance is not a fundamental symmetry, but it only appears approximately for sufficiently fine triangulations.

Recently it has been demonstrated that a Poincare 2-group state-sum model can describe a quantum GR theory, and that the corresponding state sum can be reduced to a Regge state-sum model [5, 6]. The 2-group state sum models, which are generalizations of the spin foam models, were introduced in order to solve one of the key problems of spin foam models, and that is to demonstrate the existence of the semiclassical limit which corresponds to GR with small quantum corrections. The semiclassical limit of a spin foam model can be studied by using the effective action (EA) method [7, 8], and as argued in [6], the classical limit of a spin foam model effective action is the area-Regge action. It is not difficult to extend the effective action method to a Poincare 2-group state-sum model [6], which can be reduced to a Regge

state-sum (RSS) model. The study of the effective action for a RSS model has started in [6], but there the effective action was defined by the equation where the positivity of the edge lengths was not imposed. The goal of this paper is to study the semiclassical expansion of the effective action for a Regge state-sum model when the positivity constraint is imposed.

In section 2 we define a Regge state-sum model and study the corresponding effective action equation for power-law and exponential measures when the positivity of the edge lengths is imposed in the one-dimensional case. In section 3 we study the same problem in a higher-dimensional case, and derive the conditions for the validity of the semiclassical approximation. In section 4 we show that a generalization of an exponential measure can generate the cosmological constant term in the effective action. In section 5 we study the case when the edge lengths take a discrete set of values and in section 6 we present our conclusions. We also include two appendices with relevant formulas for the error function and Gaussian sums.

2. Regge state-sum models

We will define a Regge state-sum model by fixing a spacetime triangulation $T(M)$ of a compact 4-manifold M . The triangulation $T(M)$ will be a four-dimensional simplicial complex, and we will label the edges ϵ of $T(M)$ with non-negative numbers L_ϵ , which will be called the edge lengths, since L_ϵ for a triangle Δ of $T(M)$ will satisfy the triangle inequalities. In order to completely specify a Regge state-sum model, we have to specify the set of values of L_ϵ . The simplest choice is the interval $[0, \infty)$, but one can also consider $[a, \infty)$ and $[a, b]$ intervals, where $a, b > 0$. We will also consider the case of a discrete set of values, given by $L_\epsilon = l_0 n$, $n \in \mathbf{N}$ and $l_0 > 0$.

Let $L_\epsilon \in [0, \infty)$. The corresponding state sum can be defined by the following integral

$$Z = \int_{D_E} \prod_{\epsilon=1}^E \mu(L_\epsilon) dL_\epsilon \exp \left(i S_R(L) / l_P^2 \right), \quad (1)$$

where

$$S_R = \sum_{\Delta=1}^F A_\Delta(L) \theta_\Delta(L),$$

is the Regge action (A_Δ is the area of Δ and θ_Δ is the deficit angle) and $l_P^2 = G_N \hbar$ is the Planck length. The integration region D_E is a subset of

\mathbf{R}_+^E where the triangle inequalities hold. The reason that in (1) appears the Planck length is that the GR action S_{GR} is given by S_R/G_N , so that $S_{GR}/\hbar = S_R/l_P^2$.

The measure μ has to be chosen such that Z is convergent and that the corresponding effective action allows a semiclassical expansion and has the correct classical limit. We will study this problem for a class of measures such that

$$\mu(L) \approx (L/L_1)^{-p} e^{-(L/L_0)^\alpha} \quad (2)$$

for L large, where L_0, L_1, p and α are parameters to be determined. We will restrict α to be non-negative number, so that when $\alpha > 0$ we will have a finite Z . When $\alpha = 0$, we will assume that p is such that Z is finite, see [6].

The effective action $\Gamma(L)$ can be defined by using an integro-differential equation from QFT, see [9, 8]. However, in the QG case the “field variable” L_ϵ does not take all the values from \mathbf{R} , but $L_\epsilon \geq 0$. In order to see the difference it is sufficient to consider the case of a single variable $L \in [a, b]$. Let us start from the generating functional

$$Z(J) = \int_a^b dL \mu(L) \exp \left(\frac{i}{\hbar} S(L) + iJL \right) = e^{\frac{i}{\hbar} W(J)},$$

where $S(L)$ is a C^∞ function. We define the “classical field” as

$$\bar{L} = \frac{1}{\hbar} W'(J),$$

where $W(J) = \log Z(J)$.

The corresponding Legendre transformation is given by

$$\Gamma(\bar{L}) = W(J) - \hbar J \bar{L},$$

so that we obtain

$$e^{i\Gamma(\bar{L})/\hbar} = \int_{a-\bar{L}}^{b-\bar{L}} dl \mu(\bar{L} + l) \exp \left(\frac{i}{\hbar} [S(\bar{L} + l) - \Gamma'(\bar{L}) l] \right), \quad (3)$$

where $l = L - \bar{L}$ is the “quantum fluctuation”. In the QFT case $a = -\infty$ and $b = \infty$, which gives the QFT equation

$$e^{i\Gamma(L)/\hbar} = \int_{-\infty}^{\infty} dl \mu(L + l) \exp \left(\frac{i}{\hbar} [S(L + l) - \Gamma'(L) l] \right), \quad (4)$$

while in the simplest QG case we have $a = 0$ and $b = \infty$, so that

$$e^{i\Gamma(L)/\hbar} = \int_{-L}^{\infty} dl \mu(L+l) \exp\left(\frac{i}{\hbar}[S(L+l) - \Gamma'(L)l]\right). \quad (5)$$

In the QG case $S(L) = S_R(L)$ and we will look for a perturbative solution

$$\Gamma(L) = S(L) + \hbar\Gamma_1(L) + \hbar^2\Gamma_2(L) + \dots \quad (6)$$

for $L \rightarrow \infty$ of the equation (5) with a measure (2). The appearance of a semi-infinite interval of integration in the QG case may change the nature of the perturbative solution. Namely, the QFT perturbative expansion is based on the Gaussian integration formula

$$\int_{-\infty}^{\infty} e^{-zx^2/\hbar - wx} dx = \sqrt{\frac{\pi\hbar}{z}} e^{\hbar w^2/4z} = \sqrt{\pi\hbar} e^{-\frac{1}{2}\log z + \hbar w^2/4z}, \quad (7)$$

where $Re z > 0$ or $Re w \geq 0$ if $Re z = 0$. In the QG case it changes to

$$\begin{aligned} \int_{-L}^{\infty} e^{-zx^2/\hbar - wx} dx &= \sqrt{\pi\hbar} \exp\left[-\frac{1}{2}\log z + \frac{\hbar w^2}{4z}\right. \\ &\quad \left. + \frac{\sqrt{\hbar}e^{-z\bar{L}^2/\hbar}}{2\sqrt{\pi z\bar{L}}} \left(1 + O\left(\frac{\hbar}{z\bar{L}^2}\right)\right)\right], \end{aligned} \quad (8)$$

where $\bar{L} = L + \hbar w/2z$, see the Appendix A.

The key difference between (7) and (8) is the appearance of the non-analytic term in \hbar in (8), which is given by $\sqrt{\hbar}e^{-z\bar{L}^2/\hbar}$. Hence, if this non-analytic term is not suppressed for large L , we will not have a semiclassical solution for large L . Therefore we need

$$\lim_{L \rightarrow \infty} Re(z\bar{L}^2/\hbar) = +\infty,$$

where

$$z\bar{L}^2/\hbar = L^2z + Lw + (w^2/4z)\hbar.$$

In the QG case we can replace \hbar with l_P^2 and

$$z/l_P^2 = -iS''(L)/2l_P^2 - p/L^2 + \alpha(\alpha-1)(L_1)^{-\alpha}L^{\alpha-2},$$

while

$$w = -i(\Gamma'_1 + l_P^2\Gamma'_2 + \dots).$$

Let us assume that α and p have such values that allow a perturbative solution. In the next section we will show that in the perturbative case

$$\Gamma_1 = ip \ln(L/L_0) + i(L/L_1)^\alpha + \frac{i}{2} \log S'''(L),$$

and

$$\Gamma_{n+1}(L) = O(L^{n(\alpha-2)}).$$

Hence

$$Re(z \bar{L}^2/l_P^2) = \alpha^2(L/L_1)^\alpha + LS'''(L)/2S''(L) + O(L^{\alpha-2}) + O(L^{2\alpha-4}) + O(L^{3\alpha-6}) + \dots.$$

If $0 < \alpha < 2$, then

$$Re(z \bar{L}^2/l_P^2) \approx \alpha^2(L/L_1)^\alpha,$$

when $L \rightarrow \infty$, since $LS'''(L)/S''(L) = O(1)$ due to $S(L) = O(L^2)$, so that the non-perturbative terms will be exponentially suppressed. In this case we can use the QFT equation (4) to obtain $\Gamma_n(L)$ for large L .

If $\alpha \geq 2$, we cannot say what is the large L asymptotics of $Re(z \bar{L}^2/l_P^2)$. However, in the next section we will see that it is possible to answer this question by studying the structure of the perturbative series (6).

If $\alpha = 0$, then

$$Re(z \bar{L}^2/l_P^2) \approx LS'''(L)/2S''(L),$$

which is a bounded oscillating function. This implies that the non-perturbative terms will not be suppressed for large L , so that the semiclassical solution does not exist for any value of p . This is quite surprising given that the perturbative solution exists for any p in the QFT case.

3. Higher-dimensional case

Let us now try to generalize the analysis of the previous section to the case $E > 1$. Let $L = (L_1, \dots, L_E) \in D_E$, then we obtain the following integro-differential equation

$$e^{i\Gamma(L)/l_P^2} = \int_{D_E(L)} \prod_{\epsilon=1}^E \mu(L_\epsilon + l_\epsilon) dl_\epsilon \exp \left(iS_R(L + l)/l_P^2 - i \sum_{\epsilon} \frac{\partial \Gamma}{\partial L_\epsilon} l_\epsilon / l_P^2 \right), \quad (9)$$

where the integration region $D_E(L)$ is a subset of \mathbf{R}^E obtained by translating D_E by a vector $-L$.

The main problem with generalizing the $E = 1$ results is that we do not know how to calculate exactly the integral

$$I_0 = \int_{D_E(L)} d^E l \exp \left(-\langle l, zl \rangle / l_P^2 + \langle w, l \rangle \right),$$

where z is a $E \times E$ symmetric complex matrix and $\langle x, y \rangle = \sum_{k=1}^E x_k y_k$. A reasonable conjecture is that for large L

$$I_0 \approx \int_{C_E(L)} d^E l \exp \left(-\langle l, zl \rangle / l_P^2 + \langle w, l \rangle \right),$$

where

$$C_E(L) = [-L_1, \infty) \times \cdots \times [-L_E, \infty).$$

From this conjecture it follows that

$$I_0 \approx \left(\frac{\pi l_P^2}{4} \right)^{E/2} (\det z)^{-1/2} e^{l_P^2 \langle w, z^{-1} w \rangle / 4} \prod_k \left[1 + \operatorname{erf} \left(\frac{\tilde{L}_k \sqrt{\lambda_k}}{l_P} + \frac{l_P \tilde{w}_k}{2\sqrt{\lambda_k}} \right) \right],$$

where λ_k are the eigenvalues of the matrix z and $\tilde{w} = Uw$, where U is the matrix which puts z in the diagonal form, i.e. $z = U^{-1} \operatorname{diag}(\lambda_1, \dots, \lambda_E) U$.

Then a semiclassical solution of (9) will exist for the measures satisfying (2) when $L \gg l_P$ and $0 < \alpha < 2$. As in the $E = 1$ case, we cannot determine what happens for $\alpha \geq 2$, so that we have to use a different method. Let us assume that α and p are such that the perturbative expansion

$$\Gamma(L) = S_R(L) + l_P^2 \Gamma_1(L) + l_P^4 \Gamma_2(L) + \cdots \quad (10)$$

is valid. We will try to derive some restrictions on α and p from the requirement that

$$\frac{l_P^2 |\Gamma_{n+1}(L)|}{|\Gamma_n(L)|} \ll 1 \quad (11)$$

for $L/l_P \gg 1$ and all n . The requirement (11) defines the semiclassical expansion, since it implies that the quantum corrections are much smaller than the classical value. A weaker version of (11) is

$$\frac{l_P^2 |\Gamma_{n+1}(L)|}{|\Gamma_n(L)|} < 1$$

for $L/l_P > 1$ and all n , and in this case we will consider the solution (10) to be perturbative.

Let $\alpha > 0$ and if the perturbative expansion is valid we can use the approximation $D_E(L) \approx \mathbf{R}^E$ to solve the equation (9). Then

$$\Gamma_1(L) = i \sum_{\epsilon=1}^E [(L_\epsilon/L_0)^\alpha + p \ln(L_\epsilon/L_1)] + \frac{i}{2} \text{Tr} \log S_R''(L), \quad (12)$$

which is of $O(L^\alpha)$. The higher-order corrections will be given by the same diagrams as in the QFT case, plus the corrections coming from the measure factor. These corrections can be evaluated by using the n -loop effective action diagrams (EAD) whose k -valent vertices ($k \geq 3$) carry the weights $\bar{S}_k = i\bar{S}_R^{(k)}(L)/k!$ and the edges carry the propagator $\bar{G}(L) = i(\bar{S}_R'')^{-1}$, where

$$\bar{S}_R = S_R + i l_P^2 \sum_{\epsilon=1}^E [(L_\epsilon/L_0)^\alpha + p \ln(L_\epsilon/L_1)] .$$

This follows from the EA equation (9), since it can be rewritten as the EA equation with trivial measure term and the action \bar{S} . The perturbative solution will take the form

$$\Gamma = \bar{S} + l_P^2 \bar{\Gamma}_1 + l_P^4 \bar{\Gamma}_2 + \dots ,$$

where $\bar{\Gamma}_n$ will be given by the EAD with \bar{G} propagator and \bar{S}_k vertices. Since

$$\bar{\Gamma}_n = \Gamma_{n,0} + l_P^2 \bar{\Gamma}_{n,1} + l_P^4 \bar{\Gamma}_{n,2} + \dots ,$$

we obtain

$$\Gamma = S + l_P^2 (-i \log \mu + \Gamma_{1,0}) + l_P^4 (\Gamma_{2,0} + \bar{\Gamma}_{1,1}) + l_P^6 (\Gamma_{3,0} + \bar{\Gamma}_{1,2} + \bar{\Gamma}_{2,1}) + \dots . \quad (13)$$

For example

$$\Gamma_2 = \langle (S_3)^2 G^3 \rangle + \langle S_4 G^2 \rangle + \text{Res} [l_P^{-4} \text{Tr} \log \bar{G}]$$

$$\Gamma_3 = \langle (S_3)^4 G^6 \rangle + \langle S_3 S_4 G^4 \rangle + \langle S_6 G^3 \rangle + \text{Res} [l_P^{-6} (\text{Tr} \log \bar{G} + \langle (\bar{S}_3)^2 \bar{G}^3 \rangle + \langle \bar{S}_4 \bar{G}^2 \rangle)]$$

and so on. Here $\langle XY \dots \rangle$ denotes the sum of all possible contractions of the tensors X, Y, \dots , which is given by the corresponding EAD, while

$$\text{Res} (z^{-n} f(z)) = \frac{f^{(n-1)}(0)}{(n-1)!} ,$$

where $z = l_P^2$.

By using that $S_n = O(L^{2-n})$ and $\bar{S}_n = O(L^{2-n}) + O(L^{\alpha-n})^1$, we obtain

$$\Gamma_{n+1,0} = O(L^{-2n}), \quad \bar{\Gamma}_{n+1-k,k} = O(L^{k\alpha-2n}),$$

where $k = 1, 2, \dots, n$. Consequently

$$l_P^{2n} \Gamma_{n+1,0} = O((l_P/L)^{2n}), \quad l_P^{2n} \bar{\Gamma}_{n+1-k,k} = O((L/L_0)^{k\alpha} (l_P/L)^{2n}).$$

Hence

$$\begin{aligned} l_P^{2n} \Gamma_{n+1}(L) &= O\left((l_P/L)^{2n}\right) + O\left((l_P/L)^{2n} (L/L_0)^{n\alpha}\right) \\ &= O\left((l_P/L)^{2n}\right) + O\left((L/L_s)^{n(\alpha-2)}\right), \end{aligned}$$

where

$$L_s = \left(L_0^\alpha / l_P^2\right)^{\frac{1}{\alpha-2}}, \quad (14)$$

is a new length scale which together with l_P will determine the validity of the semiclassical expansion.

By using the criterion (11) we obtain that the perturbative expansion (13) will be semiclassical if

$$l_P^2/L^2 \ll 1, \quad (L/L_s)^{\alpha-2} \ll 1. \quad (15)$$

The condition (15) will be satisfied if $L \gg l_P$ and $L \gg L_s$ for $\alpha < 2$, while for $\alpha > 2$ we need that $l_P \ll L \ll L_s$.

When $\alpha = 2$, we have

$$l_P^{2n} \Gamma_{n+1}(L) = O\left((l_P/L)^{2n}\right) + O\left((l_P/L_0)^{2n}\right),$$

so that the series (13) will be semiclassical for $L_0 \gg l_P$ and $L \gg l_P$.

Note that the Γ_1 takes imaginary number values, and the higher-order quantum corrections Γ_n will in general take complex number values. The same happens in QFT, and since we want that the effective action is real, we have to restrict our complex solution to real values. In QFT this is done via the Wick rotation. However, in QG there is no a background Minkowski metric, so that the equivalent of the Wick rotation is

$$\Gamma \rightarrow \text{Re } \Gamma(L) \pm \text{Im } \Gamma(L), \quad (16)$$

¹We define $f(x_1, x_2, \dots, x_n) = O(x^\alpha)$ if $f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) \approx \lambda^\alpha g(x_1, x_2, \dots, x_n)$ for $\lambda \rightarrow \infty$.

see [7]. The sign ambiguity can be fixed by an experimental input, see section 6.

4. Cosmological constant measures

When $\alpha > 2$ the expansion (13) will be semiclassical if $l_P \ll L \ll L_s$, so that we need that $L_s \gg l_P$, which is satisfied if $L_0 \gg l_P$. The interesting case is $\alpha = 4$, because this type of measure can be related to the cosmological constant term. Let us consider the following PI measure

$$\mu_c(L) = \exp \left(- \sum_{\sigma} V_{\sigma}(L)/L_0^4 \right) = \exp \left(-V_M(L)/L_0^4 \right), \quad (17)$$

where $V_M(L)$ is the 4-volume of $T(M)$ and σ is a 4-simplex in $T(M)$, while $V_{\sigma}(L)$ is the 4-volume of σ . The measure μ_c is a slight generalization of the usual PI measure, since

$$\mu_c(L) = \prod_{\sigma} \mu(V_{\sigma}) \neq \prod_{\epsilon} \mu(L_{\epsilon}).$$

Given that $V_{\sigma}(L) = O(L^4)$ for large L , we have

$$\log \mu_c(L) = O((L/L_0)^4),$$

so that the validity of the semiclassical approximation will be the same as in the case of the $\alpha = 4$ measure (2) with $p = 0$.

If we define the effective action as $S_{eff} = (Re \Gamma \pm Im \Gamma)/G_N$ we will obtain

$$S_{eff} = \frac{S_R}{G_N} \pm \frac{l_P^2}{G_N L_0^4} V_M \pm \frac{l_P^2}{2G_N} Tr \log S_R'' + O(l_P^4). \quad (18)$$

Hence the second term can be interpreted as the cosmological constant term with the value of the cosmological constant given by

$$\Lambda = \mp \frac{l_P^2}{2L_0^4} = \mp \frac{1}{2L_s^2}. \quad (19)$$

Note that the value (19) will be very small in units of l_P^{-2} , since

$$l_P^2 |\Lambda| = \frac{1}{2} \left(\frac{l_P}{L_0} \right)^4$$

and $l_P/L_0 \ll 1$. Therefore we have a mechanism to generate a small cosmological constant from the PI measure, as a first-order quantum correction. If we define $L_\Lambda = 1/\sqrt{\Lambda}$, then the observed value for Λ gives $L_\Lambda \approx 10^{26} m$. Since $L_s = L_\Lambda/\sqrt{2}$, we get $L_0 \approx 1 mm$ so that $l_P/L_0 \approx 10^{-32}$.

Observe that $\Gamma_3(L) = O(L^4)$, so that one can have an $O(l_P^6/L_0^8)$ correction to the CC value (19). Hence the exact value of the cosmological constant will be

$$\Lambda = \mp \frac{l_P^2}{2L_0^4} \left(1 + \text{const.} \frac{l_P^4}{L_0^4} \right). \quad (20)$$

However, the third-order quantum correction to Λ is of $O(l_P^4/L_0^4)$ of its value, which is negligible in the case of the observed value of Λ .

5. Discrete-length Regge models

Let us now analyze the case of the discrete spectrum of L_ϵ . Let $L_\epsilon = \gamma n_\epsilon l_P$ where $n \in \mathbf{N}$ and $\gamma > 0$. Then

$$Z = \sum_{n \in N_{\gamma,E}} \prod_{\epsilon} e^{-(\gamma n_\epsilon/\beta)^\alpha} \exp \left(i S_R(L(n))/l_P^2 \right),$$

where $N_{\gamma,E}$ is a subset of \mathbf{N}^E such that $\gamma n l_P \in D_E$.

The effective action equation is given by

$$e^{i\Gamma(L)/l_P^2} = \sum_{n \in N_{\gamma,E}} \prod_{\epsilon} \mu(L_\epsilon + l_\epsilon(n)) \exp \left(i S_R(L + l(n))/l_P^2 - i \sum_{\epsilon} \frac{\partial \Gamma}{\partial L_\epsilon} l_\epsilon(n)/l_P^2 \right), \quad (21)$$

where $l_\epsilon(n) = \gamma l_P n_\epsilon - L_\epsilon$. We will restrict the variable L in (21) to $L_\epsilon = \gamma l_P n_\epsilon^0$, so that $l_\epsilon = \gamma l_P m_\epsilon$ where $m_\epsilon \in \mathbf{Z}$.

One expects to obtain the same result as in the continuous case. However, there is an obstruction, due to the fact that

$$\sum_{m=-k}^{\infty} f(m) \neq \int_{-k}^{\infty} f(l) dl.$$

In our case this problem appears when computing the one-loop correction, which is given by the logarithm of

$$\sum_{m \in \mathbf{Z}^E} \exp \left(\frac{i}{2} \langle \gamma m, S_R''(L) \gamma m \rangle \right).$$

Since

$$\sum_{m \in \mathbf{Z}} \exp(iam^2) \neq \sqrt{\frac{i\pi}{a}},$$

we cannot use the Gaussian integral approximation. However, one can show that

$$\sum_{m \in \mathbf{Z}} \exp(iam^2) \approx \sqrt{\frac{i\pi}{a}},$$

for $a \rightarrow 0$ (see the Appendix B). Hence

$$\sum_{m \in \mathbf{Z}^E} \exp\left(\frac{i}{2} \langle m, \gamma^2 S_R''(L) m \rangle\right) \approx (2i\pi)^{E/2} (\det(\gamma^2 S_R''(L)))^{-1/2}, \quad (22)$$

only if the entries of the Hessian matrix $S_R''(L)$ satisfy

$$\gamma^2 |S_R''(L)| \ll 1. \quad (23)$$

Since $S_R''(L) = O(1)$, we need $\gamma^2 \ll 1$ which implies $\gamma \ll 1$.

Therefore the semiclassical approximation will be valid only if the spectrum gap is much smaller than l_P . This is a surprising result, since it implies that in the natural case when the spectrum gap is of order l_P , which corresponds to $\gamma \approx 1$, one cannot solve the EA equation (21) perturbatively. Even if we abandon the positivity of $L + l$, and replace $D_E(L)$ with \mathbf{R}^E , the result (23) holds.

The requirement (23) can be also applied to the semiclassical approximation of the effective action for spin foam models. In the spin foam case, instead of the edge lengths, we have the triangle area variables $j l_P^2$, such that $j \in (\mathbf{N}/2)^F$ and $S_R(L)/l_P^2 \rightarrow S(j)$ where

$$S(j) \approx \sum_{f=1}^F j_f \theta_f(j),$$

for $j_f \gg 1$, see [7]. Hence $\gamma = 1/2$ in the spin foam case. However, there is no problem for the semiclassical approximation, since the Hessian satisfies $S''(j) = O(1/j)$ and therefore $|S''(j)| \ll 1$ so that

$$\sum_{m \in \mathbf{Z}^F} \exp\left(\frac{i}{8} \langle m, S''(j) m \rangle\right) \approx (8i\pi)^{F/2} (\det(S''(j)))^{-1/2}.$$

6. Conclusions

A standard PI measure $\mu(L) \approx (L/L_0)^p$ for $L \rightarrow \infty$ does not allow a semi-classical effective action if we impose the positivity of L in the effective action equation. However, an exponential PI measure $\mu(L) = e^{-(L/L_0)^\alpha}$ with $\alpha > 0$ allows a semiclassical solution provided that the restrictions (15) are satisfied.

Note that if one defines the effective action equation with the exponential measure such that the integration region in (9) is \mathbf{R}^E instead of $D_E(L)$, one will still obtain the same perturbative solution as in the $D_E(L)$ case and the domain of validity of the perturbative solution will be the same. The advantage of using a simpler integration region is that the effective action equation is simpler to solve. However, the price is that $S_R(L+l)$ can take complex number values. This also happens for the effective action equation used in [7, 8, 6], where \mathbf{R}_+^E was chosen as the integration region. As argued in [6], this is not a problem, since even when $S_R(L+l)$ takes only real values, one still obtains a complex-valued perturbative solution $\Gamma(L)$. Hence in the \mathbf{R}^E case one is not forced to use the exponential measures. However, these measures still have an advantage of being able to generate the cosmological constant term.

When the number of the edges E is large, one can make the smooth manifold approximation of $\Gamma_n(L)$ terms. In that case

$$S_R(L) \approx \int_M d^4x \sqrt{|\det g|} R_g,$$

while for the $\log \mu$ term in Γ_1 we have

$$\sum_{\epsilon=1}^E f(L_\epsilon/L_0) \approx \int_M d^4x \sqrt{|\det g|} \mathcal{F}_f[g(x)],$$

where $f(x) = x^\alpha$ or $\ln x$. However, we do not know what is the functional \mathcal{F}_f , except in the case of the cosmological constant measure (17), where the smooth-manifold approximation is simply the 4-volume term.

As far as the trace-log term is concerned, its continuum approximation can be obtained by using the effective field theory approach, see [10, 11, 12]. If we introduce a cut-off scale L_c such that $L \geq L_c \gg l_P$ then

$$Tr(\log S_R'') = \sum_{\epsilon} (\log S_R''(L))_{\epsilon\epsilon} \approx \int_M d^4x \sqrt{g} \left[a(L_c) R^2 + b(L_c) R_{\mu\nu} R^{\mu\nu} \right],$$

where the functions $a(L_c)$ and $b(L_c)$ will be given by the cut-off regularization of the corresponding QFT diagrams. Note that L_c can be chosen to be the minimal distance for which we know that perturbative QFT is applicable. From the LHC experiments we know that $L_c \leq 10^{-20} m$.

An exciting developement is that the measure (17) can generate the cosmological constant term. We obtain an exact formula (20) for the value of cosmological constant, which is practically the same as the first-order approximation (19), since the semiclassical approximation requires that $L_0 \gg l_P$. This also insures that the corresponding cosmological constant will take a very small value in the units of l_P^{-2} . Hence we have a mechanism to generate a very small cosmological constant as a quantum gravity effect. It remains to be seen what is the contribution of the matter sector to the CC value. If the matter contribution is for some reason small or zero, one would have a theory with a naturally small cosmological constant. The matter contribution to the cosmological constant will also resolve the sign ambiguity in (16), since we know that the observed CC value is positive.

Another surprising result of our approach is that the validity of the semiclassical approximation in the case of discrete L requires that the spectrum gap is much smaller than l_P , since the entries of the Hessian matrix satisfy $S_R''(L) = O(1)$. Hence the natural case where L_ϵ is an integer multiple of l_P requires a nonperturbative solution of the EA equation. In the case of spin foam models, the triangle area is similarly an integer multiple of l_P^2 for large spins, but there is no problem with the semiclassical approximation since the Hessian satisfies $S''(j) = O(1/j)$ for large spins j , so that the Gaussian sums can be approximated with Gaussian integrals.

The coupling of matter can be studied by replacing S_R/l_P^2 in the EA equation by

$$S_R(L)/l_P^2 + S_{Rm}(L, \phi, \psi, A)/\hbar = [S_R(L) + G_N S_{Rm}(L, \phi, \psi, A)]/l_P^2,$$

where S_{Rm} is the sum of Regge actions for scalars ϕ , fermions ψ and gauge fields A .

The effective action equation can be solved perturbatively for $L_\epsilon \gg l_P$, which is the semiclassical regime. An important problem is how to solve the EA equation for $L_\epsilon \approx l_P$, which is the deep quantum regime. Since there the perturbation theory fails, one has to find an alternative method. A promissing approach is to use the fact that the EA is also the generating

functional for the one-particle-irreducible (1PI) Green's functions, so that

$$\Gamma(L) = \sum_{\epsilon, \epsilon'} \tilde{\Gamma}_2(\epsilon, \epsilon') L_\epsilon L_{\epsilon'} + \sum_{\epsilon, \epsilon', \epsilon''} \tilde{\Gamma}_3(\epsilon, \epsilon', \epsilon'') L_\epsilon L_{\epsilon'} L_{\epsilon''} + \dots, \quad (24)$$

where $\tilde{\Gamma}_n(\epsilon)$ is the 1PI part of the n -point Green's function

$$G(\epsilon_1, \dots, \epsilon_n) = \frac{1}{Z} \int_{D_E} d^E L \mu_T(L) L_{\epsilon_1} \dots L_{\epsilon_n} e^{iS_R(L)/l_P^2},$$

where $\mu_T = \prod_\epsilon \mu(L_\epsilon)$ or $\mu_T = \prod_\sigma \mu(V_\sigma)$. This integral can be calculated numerically, which can then give the expansion (24). One can also study the expansion (24) by using the mini-superspace approximation $L_1 = \dots = L_E = L$ and the corresponding EA equation.

Also note that for the exponential measures with $\alpha > 2$ there will be a maximal length L_s for which the semiclassical approximation is valid, so that in this case there may be non-perturbative quantum effects at large distances $L \approx L_s$.

The fact that there is a minimal (l_P) and a maximal (L_s) length for which the semiclassical approximation is valid, raises the question of the relation of l_P and L_s to the minimal and the maximal length in the spectrum of L . Formally, one can choose any interval $[a, b]$ for L_ϵ of the state-sum model, where $0 \leq a < b$. In the case $a > l_P$ and $b < L_s$ one will have a QG theory with purely perturbative QG effects. However, a more interesting case is $a \leq l_P$ and $b \geq L_s$, since in such a theory one can have non-perturbative QG effects for small and large distances.

Note that the knowledge of the effective action is not sufficient for a complete QG theory. The concept of an effective action only makes sense for spacetimes whose topology is $\Sigma \times [0, 1]$, where Σ is a 3-manifold. We also need a wavefunction which can be associated to a cup-manifold $C(\Sigma)$, where $C(\Sigma)$ is a compact 4-manifold whose boundary is Σ . This is essentially the Hartle-Hawking wavefunction [13], and it would be interesting to develop a state-sum quantum cosmology theory based on these concepts.

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Appendix A

Consider the following integro-differential equation

$$e^{i\Gamma(L)/\varepsilon} = \int_{-L}^{\infty} dl e^{i[S(L+l)-\Gamma'(L)l]/\varepsilon},$$

where $S(L)$ is a C^∞ function, $L > 0$ and ε is a small parameter. We want to solve it perturbatively in ε as

$$\Gamma(L) = S(L) + \sum_{n>0} \varepsilon^n \Gamma_n(L),$$

up to an additive constant.

Since

$$S(L+l) = S(L) + \sum_{n>0} S_n(L) l^n,$$

were $S_n(L) = S^{(n)}(L)/n!$, we obtain

$$\Gamma_1 + \varepsilon \Gamma_2 + \varepsilon^2 \Gamma_3 + \dots = (-i) \log \int_{-L}^{\infty} dl \exp \left[\frac{i}{\varepsilon} S_2 l^2 - i \bar{\Gamma}'_1 l + \frac{i}{\varepsilon} \sum_{n>2} S_n l^n \right], \quad (A.1)$$

where $\bar{\Gamma}_1 = \Gamma_1 + \varepsilon \Gamma_2 + \varepsilon^2 \Gamma_3 + \dots$.

The integral in (A.1) is of the type

$$I = \int_{-L}^{\infty} dl e^{-zl^2 + wl} \exp \left(\sum_{n>2} s_n l^n \right),$$

which we rewrite as

$$I = \int_{-L}^{\infty} dl e^{-zl^2 + wl} \left(1 + \sum_{n>2} \hat{s}_n l^n \right).$$

Hence we will need the integrals

$$I_n = \int_{-L}^{\infty} dl e^{-zl^2 + wl} l^n,$$

which can be calculated by differentiating I_0 wrt w . It is easy to show that

$$I_0 = \sqrt{\frac{\pi}{4z}} e^{\frac{w^2}{4z}} \left[1 + \operatorname{erf} \left(L\sqrt{z} + \frac{w}{2\sqrt{z}} \right) \right],$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The domain of the error function can be extended to any complex number z by using the Taylor expansion

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}.$$

For large x we can use

$$\operatorname{erf}(x) = 1 + \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 + \sum_{n=1}^{N-1} \frac{(-1)^n (2n-1)!!}{2^n x^{2n}} + R_N(x) \right), \quad (A.2)$$

where $R_N(x) = O(x^{-2N})$. If x takes complex values, we can use (A.2) for large $|x|$ and $|\arg(x)| < 3\pi/4$ [15]. For $|\arg(x)| < \pi/2$

$$R_N(x) = \frac{(-1)^N (2N-1)!!}{2^N x^{2N}} \theta,$$

where

$$\theta = \int_0^{\infty} e^{-t} (1 + t/x^2)^{-N-1/2} dt.$$

For $|\arg(x)| < \pi/4$ one has $|\theta| < 1$, see [15].

Appendix B

Let

$$S(a) = \sum_{n=0}^{\infty} e^{-an^2},$$

where $a > 0$. It was shown in [14] that as $a \rightarrow 0$

$$S(a) = \sqrt{\frac{\pi}{4a}} + \frac{1}{2} e^{-a/4} \left[\frac{\sinh \sqrt{a}}{\sqrt{a}} - \sum_{n=0}^N c_n a^{n+1/2} H_{2n+1}(\sqrt{a}/2) \right] + O(a^{N+3/2}),$$

where

$$c_n = \frac{(2^{2n+1} - 1) B_{2n+2}}{2^{2n} (2n+2)!},$$

B_n are Bernoulli numbers and $H_n(x)$ are Hermite polynomials.

If $a < 1$, then in the limit $N \rightarrow \infty$ we obtain

$$S(a) = \sqrt{\frac{\pi}{4a}} + \frac{1}{2} e^{-a/4} \left[\frac{\sinh \sqrt{a}}{\sqrt{a}} - \sum_{n=0}^{\infty} c_n a^{n+1/2} H_{2n+1}(\sqrt{a}/2) \right].$$

Let $R(a) = S(a) - \sqrt{\frac{\pi}{4a}}$, then

$$R(a) = \sum_{n=0}^{\infty} r_n a^{n/2},$$

for $a < 1$. We can now define a complex function

$$R(z) = \sum_{n=0}^{\infty} r_n z^{n/2},$$

for $|z| < 1$. Consequently we can define

$$S(z) = \sqrt{\frac{\pi}{4z}} + R(z),$$

for $0 < |z| < 1$, so that

$$S(-ia) \approx \sqrt{\frac{i\pi}{4a}}$$

as $a \rightarrow 0$.

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